Identifiability and non-identifiability in acoustic tomography of moving fluid

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Abstract

We consider a model time-harmonic wave equation of acoustic tomography of moving fluid in an open bounded domain in $\mathbb{R}^d$, $d \geq 2$, with variable sound speed c, density ρ, fluid velocity v and absorption coefficient α. We give global uniqueness results for related inverse boundary value problem for the cases of boundary measurements given for two and for three fixed frequencies. Besides, we also give a non-uniqueness result for this inverse problem for the case of boundary measurements given for all frequencies.

Keywords: moving fluid, magnetic Schrödinger equation, acoustic tomography, inverse boundary value problems, identifiability and non-identifiability

AMS Classification: 35R30, 35Q35

1 Introduction

We consider a moving fluid in an open bounded domain $D \subset \mathbb{R}^d$ with sound speed $c = c(x)$, density $\rho = \rho(x)$, fluid velocity vector $v = v(x)$ and the sound wave absorption coefficient $\alpha = \alpha(x, \omega)$ at fixed frequency ω, where $x \in \bar{D} = D \cup \partial D$. For this fluid we consider the following model equation for the time-harmonic ($e^{-i\omega t}$) acoustic pressure ψ:

$$
\begin{gathered}
L_\omega \psi = 0 \quad \text{in } D, \\
L_\omega = -\Delta_x - 2iA_\omega(x)\nabla_x - U_\omega(x), \quad x = (x_1, \ldots, x_d) \in D,
\end{gathered} \tag{1.1}
$$

where

$$
\Delta_x = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}, \quad \nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right), \tag{1.2}
$$

$$
\begin{gathered}
A_\omega(x) = \frac{\omega v(x)}{c^2(x)} + \frac{i}{2}\frac{\nabla_x \rho(x)}{\rho(x)}, \\
U_\omega(x) = \frac{\omega^2}{c^2(x)} + 2i\omega\frac{\alpha(x,\omega)}{c(x)}, \\
\alpha(x,\omega) = \omega^{\zeta(x)}\alpha_0(x).
\end{gathered} \tag{1.3}
$$

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In different particular cases this model was considered, for example, in [AN], [BBS], [BSZR], [HN], [RBKS], [RE], [RW].

In the present work we assume that the fluid parameters c, ρ, v, α are such that

$$A_\omega \textit{ and } V_\omega \textit{ are sufficiently regular on } \bar{D} \textit{ for any } \omega > 0, \tag{1.4}$$

$$\begin{gathered} c \geq c_{min} > 0,\ \rho \geq \rho_{min} > 0,\ \alpha_0 \geq 0 \textit{ in } \bar{D} \\ \textit{for some constants } c_{min},\ \rho_{min}. \end{gathered} \tag{1.5}$$

For simplicity we consider equation (1.1) assuming that

$$\omega \notin \sigma(L_z), \tag{1.6}$$

where

$$\begin{gathered} \sigma(L_z) \textit{ consists of all } z \in \mathbb{C} \textit{ such that} \\ 0 \textit{ is a Dirichlet eigenvalue for operator } L_z \textit{ in } D. \end{gathered} \tag{1.7}$$

For equation (1.1), under assumptions (1.4), (1.6), we consider the Dirichlet-to-Neumann boundary map Λ_ω defined by the relation

$$\Lambda_\omega(\psi|_{\partial D}) = \tfrac{\partial \psi}{\partial \nu}|_{\partial D} + i(A_\omega \cdot \nu)\psi|_{\partial D}, \tag{1.8}$$

fulfiled for all sufficiently regular solutions ψ of (1.1) in $\bar{D}$, where ν is the unit exterior normal to ∂D.

We consider Λ_ω as all possible boundary measurements for the model described by equation (1.1) at fixed ω.

In the present work we consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. *Given boundary data Λ_ω for some frequencies ω, find fluid parameters c, ρ, v, α in $\bar{D}$.*

Under the assumption that

$$A_\omega \equiv 0 \quad \text{on } \partial D, \tag{1.9}$$

Problem 1.1 is closely related with the inverse scattering problem for the equation

$$L_\omega \psi = 0 \quad \text{on } \mathbb{R}^d, \tag{1.10}$$

where

$$A_\omega \equiv 0,\ \ U_\omega \equiv \frac{\omega^2}{c_0^2} \quad \text{on } \mathbb{R}^d \setminus D,$$

where c_0 can be considered as the mean value of c on ∂D. By scattering data for equation (1.10) we mean, first of all, the scattering amplitude; see, e.g., [Ag1], [AN], [HN], [No] for definitions of the scattering amplitude.

Due to results going back to [No], it is known that the inverse boundary value problems and the inverse scattering problems for operators like L_ω are, actually, equivalent. For simplicity of exposition, in the present work we do not formulate the inverse scattering version of Problem 1.1 in detail.

Note also that Problem 1.1 at fixed ω is closely related with inverse boundary value and inverse scattering problems for the Schrödinger equation in magnetic field at fixed energy. The reason is that the operator L_ω at fixed ω is closely related with the magnetic Schrödinger operator at fixed energy.

In the present work we are mainly focused on Problem 1.1 and its inverse scattering version for the case when $v \not\equiv 0$ in D, or, in other words, we are focused on the acoustic tomography of moving fluid in the framework of the wave propagation model (1.1), (1.3).

As regards results given in the literature on this acoustic tomography of moving fluid, see, e.g., [Ag1], [Ag2], [AN], [BBS], [RE], [RW] and references therein including the case when $v \equiv 0$.

In particular, in [AN] a Riemann–Hilbert problem approach to the inverse scattering version of Problem 1.1 at fixed frequency ω was developed for the case when $\rho \equiv \mathrm{const}$, $\alpha \equiv 0$, $d = 2$.

In addition, in [Ag2] global uniqueness theorems for Problem 1.1 at fixed frequency ω were proved for the case when $\rho \equiv \mathrm{const}$, $\alpha \equiv 0$, $d = 2$ or $d \geq 3$.

Note that in the present work we use, in particular, results developed in the literature for the case of the inverse boundary value problem for the Schrödinger equation in magnetic field at fixed energy; see [BS], [GuT], [KU] and references therein.

Note also that in the present work we use global uniqueness results on the Dirichlet problem for some linear and non-linear perturbations of the Laplace equation in D; see systems (3.13), (3.19) of Section 3 and related results of [GiT].

The main results of the present work can be summarized as follows:

(A1) We show that the boundary data Λ_ω given for two different frequencies $\omega = \omega_1, \omega_2$ uniquely determine the coefficients c, ρ, v under the assumptions that $\omega_1, \omega_2 \notin \sigma(L_\omega)$ and $\alpha \equiv 0$, see Theorems 2.1, 2.2 of Section 2.

(A2) We show that the boundary data Λ_ω given for three different frequencies $\omega = \omega_1, \omega_2, \omega_3$ uniquely determine c, ρ, v, α under the assumptions that $\omega_1, \omega_2, \omega_3 \notin \sigma(L_\omega)$ and ζ doesn't vanish anywhere, see Theorems 2.3, 2.4 of Section 2.

(B) We give examples of coefficients $c^{(1)}$, $\rho^{(1)}$, $v^{(1)}$, $\alpha_0^{(1)}$ and different coefficients $c^{(2)}$, $\rho^{(2)}$, $v^{(2)}$, $\alpha_0^{(2)}$ such that $\sigma(L_\omega^{(1)}) = \sigma(L_\omega^{(2)})$ and $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ for $\omega \in \mathbb{C} \setminus \sigma(L_\omega^{(i)})$ for the case when $\zeta \equiv 0$, see Theorem 2.5 of Section 2.

We recall that in the aforementioned results $\sigma(L_\omega)$ is defined by (1.7).

The uniqueness results (A1), (A2) can be considered as results on global identifiability in acoustic tomography of moving fluid, whereas the non-uniqueness result (B) can be considered as a result on principal non-identifiability in this tomographical problem.

Note also that in these results the determination of ρ is considered modulo the transformations $\rho \to C\rho$, where C is a positive constant.

The main results of the present work are presented in detail in the next section.

2 Main results

Let $W^{n,p}(D,\mathbb{C})$ denote the standard Sobolev space consisting of complex-valued functions which are n times differentiable in $L^p(D)$, where $n \geq 0$, $p \geq 1$ (including $p = \infty$) and D is the domain of Section 1. We consider also $W^{n,p}(D,\mathbb{R})$, $W^{n,p}(D,\mathbb{R}^d)$ and $W^{n,p}(D,\mathbb{C}^d)$ defined in the standard way.

We say that D is simply connected iff D is path connected and each continuous loop in D is contractible. In addition, we say that a set S in $\mathbb{R}^d$ is path connected iff each pair of points in S can be joined by a continuous path in S. See, e.g., [DNF] in connection with these definitions.

In the present work we assume mainly that

$$\begin{gathered} c \in W^{1,\infty}(D,\mathbb{R}), \ \ c > 0, \ \ \rho \in C(\bar{D}) \cup C^2(D), \ \ \rho > 0, \\ v \in W^{1,\infty}(D,\mathbb{R}^d), \ \ \alpha_0 \in C(\bar{D}), \ \ \zeta \in C(\bar{D}), \ \ \zeta \neq 0, \\ \alpha_0,\ \zeta \text{ are real-valued}, \quad \text{for } d \geq 3, \end{gathered} \tag{2.1}$$

$$\begin{gathered} c \in W^{2,p}(D,\mathbb{R}), \ \ c > 0, \ \ \rho \in W^{3,p}(D,\mathbb{R}), \ \ \rho > 0, \\ v \in W^{2,p}(D,\mathbb{R}^d), \ \ \alpha_0 \in W^{1,p}(D,\mathbb{R}), \ \ \zeta \in C(\bar{D}), \ \ \zeta \neq 0, \\ \alpha_0,\ \zeta \text{ are real-valued}, \quad \text{where } p > 2,\ d = 2. \end{gathered} \tag{2.2}$$

Let L_ω, $\sigma(L_\omega)$ and Λ_ω be defined as in Section 1.

In the present work we obtain, in particular, the following global uniqueness resutls for Problem 1.1.

Theorem 2.1. *Let D be an open bounded simply connected domain in $\mathbb{R}^d$, $d \geq 3$, with path connected C^1 boundary ∂D. Let $L_\omega^{(j)}$ and $\Lambda_\omega^{(j)}$ correspond to coefficients $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$, $\alpha^{(j)}$, where $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$ satisfy* (2.1), *$\alpha^{(j)} \equiv 0$, $j = 1, 2$. Let ω_1, $\omega_2 \in [0,+\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$, $\omega_1 \neq \omega_2$. Then the coincidence of the boundary data $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ for $\omega \in \{\omega_1, \omega_2\}$ implies that $c^{(1)} = c^{(2)}$, $\rho^{(1)} = C\rho^{(2)}$, $v^{(1)} = v^{(2)}$, where $C = const > 0$.*

Theorem 2.2. *Let D be an open bounded simply connected domain in $\mathbb{R}^2$ with path connected C^∞ boundary ∂D. Let $L_\omega^{(j)}$ and $\Lambda_\omega^{(j)}$ correspond to coefficients $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$, $\alpha^{(j)}$, where $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$ satisfy* (2.2), *$\alpha^{(j)} \equiv 0$, $j = 1, 2$. Let ω_1, $\omega_2 \in [0,+\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$, $\omega_1 \neq \omega_2$. Then the coincidence of the boundary data $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ for $\omega \in \{\omega_1, \omega_2\}$ implies that $c^{(1)} = c^{(2)}$, $\rho^{(1)} = C\rho^{(2)}$, $v^{(1)} = v^{(2)}$, where $C = const > 0$.*

Theorem 2.3. *Let D be an open bounded simply connected domain in $\mathbb{R}^d$, $d \geq 3$, with path connected C^1 boundary ∂D. Let $L_\omega^{(j)}$ and $\Lambda_\omega^{(j)}$ correspond to* *coefficients* $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$, $\alpha_0^{(j)}$, $\zeta^{(j)}$ *satisfying* (2.1), $j = 1,\ 2$. *Let* ω_1, ω_2, $\omega_3 \in (0, +\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$ *be three pairwise different frequencies. Then the coincidence of the boundary data* $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ *for* $\omega \in \{\omega_1, \omega_2, \omega_3\}$ *implies that* $c^{(1)} = c^{(2)}$, $\rho^{(1)} = C\rho^{(2)}$, $v^{(1)} = v^{(2)}$, $\alpha^{(1)} = \alpha^{(2)}$, *where* $C = const > 0$ *and* $\alpha^{(j)}(x,\omega) = \omega^{\zeta^{(j)}(x)}\alpha_0^{(j)}(x)$.

Theorem 2.4. *Let* D *be an open bounded simply connected domain in* $\mathbb{R}^2$ *with path connected* C^∞ *boundary* ∂D. *Let* $L_\omega^{(j)}$ *and* $\Lambda_\omega^{(j)}$ *correspond to coefficients* $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$, $\alpha_0^{(j)}$, $\zeta^{(j)}$ *satisfying* (2.2), $j = 1, 2$. *Let* ω_1, ω_2, $\omega_3 \in (0, +\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$ *be three pairwise different frequencies. Then the coincidence of the boundary data* $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ *for* $\omega \in \{\omega_1, \omega_2, \omega_3\}$ *implies that* $c^{(1)} = c^{(2)}$, $\rho^{(1)} = C\rho^{(2)}$, $v^{(1)} = v^{(2)}$, $\alpha^{(1)} = \alpha^{(2)}$, *where* $C = const > 0$ *and* $\alpha^{(j)}(x,\omega) = \omega^{\zeta^{(j)}(x)}\alpha_0^{(j)}(x)$.

Theorems 2.1, 2.2 and 2.3, 2.4 are proved in Sections 3, 4 and 5.

Let

$$\begin{gathered} h \textit{ be a real-valued function supported in } D, \\ h \in C^2(D) \textit{ and } |\nabla h|^2 < 1 \textit{ in } D, \end{gathered} \tag{2.3}$$

where D is an open bounded domain in $\mathbb{R}^d$.

We set

$$\begin{gathered} c^{(1)} \equiv \text{const} > 0, \quad \rho^{(1)} \equiv \text{const} > 0, \\ v^{(1)} \equiv 0, \quad \alpha_0^{(1)} \equiv \text{const} > -\tfrac{1}{2}\min_{x\in D} \Delta h(x), \end{gathered} \tag{2.4}$$

and

$$\begin{gathered} c^{(2)} = c^{(1)}(1 - |\nabla h|^2)^{-1/2}, \quad \rho^{(2)} \equiv \text{const} > 0, \\ v^{(2)} = c^{(1)}(1 - |\nabla h|^2)^{-1}\nabla h, \\ \alpha_0^{(2)} = (1 - |\nabla h|^2)^{-1/2}(\alpha_0^{(1)} + \tfrac{1}{2}\Delta h). \end{gathered} \tag{2.5}$$

Note that for these fluid parameters $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$, $\alpha_0^{(j)}$ with $\zeta^{(j)} \equiv 0$ the conditions (1.4), (1.5) are fulfiled for both cases $j = 1$ and $j = 2$.

In the present work, in addition to global uniqueness results of Theorems 2.1, 2.2, 2.3, 2.4 we give also the following non-uniqueness result.

Theorem 2.5. *Let* D *be an open bounded domain in* $\mathbb{R}^d$, $d \geq 2$, *with smooth boundary. Let* h *satisfy* (2.3), $h \not\equiv 0$, *and* $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$, $\alpha^{(j)}$, $j = 1, 2$, *be defined by* (2.4), (2.5). *Let* $L_\omega^{(j)}$ *and* $\Lambda_\omega^{(j)}$ *correspond to coefficients* $c^{(j)}$, $\rho^{(j)}$, $v^{(j)}$, $\alpha_0^{(j)}$ *with* $\zeta^{(j)} \equiv 0$. *Then* $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ *for all* $\omega \in \mathbb{C} \setminus \sigma$, *where* $\sigma = \sigma(L_z^{(1)}) = \sigma(L_z^{(2)})$.

Theorem 2.5 is proved in Section 6.

3 Proof of Theorem 2.1

Note that:

$$L_\omega^{(j)} = \sum_{k=1}^{d} \left(\frac{1}{i}\frac{\partial}{\partial x_k} + A_\omega^{(j),k} \right)^2 + q_\omega^{(j)}, \tag{3.1}$$

where

$$A_\omega^{(j)} = (A_\omega^{(j),1}, \ldots, A_\omega^{(j),d}) = \frac{\omega v^{(j)}}{(c^{(j)})^2} + \frac{i}{2}\frac{\nabla \rho^{(j)}}{\rho^{(j)}},$$

$$\begin{aligned} q_\omega^{(j)} = -\frac{\omega^2}{(c^{(j)})^2} + i\nabla \cdot \left(\frac{\omega}{(c^{(j)})^2} v^{(j)} + \frac{i}{2}\frac{\nabla \rho^{(j)}}{\rho^{(j)}} \right) - \frac{\omega^2}{(c^{(j)})^4}(v^{(j)})^2 \\ + \frac{1}{4}(\rho^{(j)})^{-2}(\nabla \rho^{(j)})^2 - \frac{i\omega v^{(j)} \nabla \rho^{(j)}}{(c^{(j)})^2 \rho^{(j)}}, \end{aligned} \tag{3.2}$$

$\nabla\cdot$ is the standard divergence, $j = 1, 2$.

By Theorem 1.1 of [BS] we have that the tangential components of the fields $A_\omega^{(1)}$ and $A_\omega^{(2)}$ on ∂D are equal. And, as a corollary,

$$\textit{tangential components of } \frac{\nabla \rho^{(1)}}{\rho^{(1)}} \textit{ and } \frac{\nabla \rho^{(2)}}{\rho^{(2)}} \textit{ on } \partial D \textit{ are equal,} \tag{3.3}$$

$$\textit{tangential components of } \frac{v^{(1)}}{(c^{(1)})^2} \textit{ and } \frac{v^{(2)}}{(c^{(2)})^2} \textit{ on } \partial D \textit{ are equal.} \tag{3.4}$$

Using (3.3) and the path connectedness of ∂D we obtain

$$\begin{aligned} \ln \rho^{(2)} - \ln \rho^{(1)} = \ln C \quad \textit{on } \partial D, \\ \rho^{(2)}|_{\partial D} = C\rho^{(1)}|_{\partial D} \quad \textit{for some positive constant } C > 0. \end{aligned} \tag{3.5}$$

Using Theorem 1.1 of [KU] and the simple connectedness of D we get:

$$q_\omega^{(2)} - q_\omega^{(1)} = 0 \quad \text{in } D \tag{3.6}$$

and

$$A_\omega^{(2)} - A_\omega^{(1)} = \nabla \varphi_\omega \quad \text{in } D, \tag{3.7}$$

where $\varphi_\omega \in W^{2,\infty}(D, \mathbb{C})$, $\omega \in \{\omega_1, \omega_2\}$.

Separating the real and the imaginary parts of (3.6) we get:

$$\begin{aligned} \omega^2 \left[\frac{1}{(c^{(1)})^2} - \frac{1}{(c^{(2)})^2} + \frac{(v^{(1)})^2}{(c^{(1)})^4} - \frac{(v^{(2)})^2}{(c^{(2)})^4} \right] \\ + \left[\left(\frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right)^2 - \left(\frac{\nabla \rho^{(1)}}{2\rho^{(1)}} \right)^2 - \nabla \cdot \left(\frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right) + \nabla \cdot \left(\frac{\nabla \rho^{(1)}}{2\nabla \rho^{(1)}} \right) \right] = 0, \end{aligned} \tag{3.8}$$

where $\omega \in \{\omega_1, \omega_2\}$;

$$\nabla \cdot \left(\frac{v^{(1)}}{(c^{(1)})^2} - \frac{v^{(2)}}{(c^{(2)})^2} \right) - \frac{\nabla \rho^{(1)}}{\rho^{(1)}} \frac{v^{(1)}}{(c^{(1)})^2} + \frac{\nabla \rho^{(2)}}{\rho^{(2)}} \frac{v^{(2)}}{(c^{(2)})^2} = 0. \tag{3.9}$$

Using (3.8) and the assumptions that ω_1, $\omega_2 \geq 0$, $\omega_1 \neq \omega_2$, we obtain

$$\left(\frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right)^2 - \left(\frac{\nabla \rho^{(1)}}{2\rho^{(1)}} \right)^2 - \nabla \cdot \left(\frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right) + \nabla \cdot \left(\frac{\nabla \rho^{(1)}}{2\rho^{(1)}} \right) = 0, \tag{3.10}$$

$$\frac{1}{(c^{(1)})^2} - \frac{1}{(c^{(2)})^2} + \frac{(v^{(1)})^2}{(c^{(1)})^4} - \frac{(v^{(2)})^2}{(c^{(2)})^4} = 0. \tag{3.11}$$

Let

$$u^{(j)} = \tfrac{1}{2}\ln\rho^{(j)}, \quad j = 1, 2. \tag{3.12}$$

Due to (3.5), (3.10) and (3.12), we have

$$\begin{cases} \Delta u^{(2)} - (\nabla u^{(2)})^2 = \Delta u^{(1)} - (\nabla u^{(1)})^2 & \text{in } D, \\ u^{(2)} = u^{(1)} + \frac{1}{2}\ln C & \text{on } \partial D. \end{cases} \tag{3.13}$$

Since $u^{(1)}$, $u^{(2)} \in C(\bar{D}) \cap C^2(D)$, it follows from Theorem 10.1 of [GiT] that

$$u^{(2)} = u^{(1)} + \tfrac{1}{2}\ln C \quad \text{in } D$$

and, consequently,

$$\rho^{(2)} = C\rho^{(1)} \quad \text{in } D. \tag{3.14}$$

Further, taking the real part of (3.7) and using (3.4) we obtain

$$\frac{v^{(2)}}{(c^{(2)})^2} - \frac{v^{(1)}}{(c^{(1)})^2} = \nabla\beta_\omega \quad \text{in } D \tag{3.15}$$

and

$$\beta_\omega = \text{const} \quad \text{on } \partial D, \tag{3.16}$$

where

$$\beta_\omega = \Re\varphi_\omega. \tag{3.17}$$

Taking into account (3.14) we define

$$a := \frac{\nabla\rho^{(1)}}{\rho^{(1)}} = \frac{\nabla\rho^{(2)}}{\rho^{(2)}}. \tag{3.18}$$

Formulas (3.9), (3.15), (3.16) and (3.18) imply that

$$\begin{cases} -\Delta\beta_\omega + a\nabla\beta_\omega = 0 & \text{in } D, \\ \beta_\omega = \text{const} & \text{on } \partial D. \end{cases} \tag{3.19}$$

Now it follows from Theorem 8.1 of [GiT] that $\beta_\omega = \text{const}$ in D. This result and formula (3.15) imply that

$$\frac{v^{(1)}}{(c^{(1)})^2} = \frac{v^{(2)}}{(c^{(2)})^2} \quad \text{in } D. \tag{3.20}$$

Finally, using (3.11), (3.20) we obtain

$$c^{(2)} = c^{(1)} \textit{ and } v^{(2)} = v^{(1)} \textit{ in } D. \tag{3.21}$$

This completes the proof of Theorem 2.1.

4 Proof of Theorem 2.2

In a similar way with the proof of Theorem 2.1, we have formulas (3.1)–(3.5) for $d = 2$.

Let

$$\mu^{(j)} = -\frac{i}{2} \ln \rho^{(j)}, \tag{4.1}$$

and

$$\widetilde{L}_\omega^{(j)} = e^{-i\mu^{(j)}} L_\omega^{(j)} e^{i\mu^{(j)}}, \tag{4.2}$$

where $e^{i\mu^{(j)}}$, $e^{-i\mu^{(j)}}$ denote the multiplication operators by the functions $e^{i\mu^{(j)}}$, $e^{-i\mu^{(j)}}$, $j = 1, 2$.

Using (4.1), (4.2) one can see that

$$\sigma(\widetilde{L}_z^{(j)}) = \sigma(L_z^{(j)}), \quad j = 1, 2, \tag{4.3}$$

and

$$\widetilde{\Lambda}_\omega^{(1)} = \widetilde{\Lambda}_\omega^{(2)}, \tag{4.4}$$

where $\widetilde{\Lambda}_\omega^{(1)}$, $\widetilde{\Lambda}_\omega^{(2)}$ are the Dirichlet-to-Neumann maps for operators $\widetilde{L}_\omega^{(1)}$, $\widetilde{L}_\omega^{(2)}$ in D, respectively, $\omega \in \{\omega_1, \omega_2\}$.

By direct computation we obtain that

$$\widetilde{L}_\omega^{(j)} = \sum_{k=1}^{2} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + \widetilde{A}_\omega^{(j),k} \right)^2 + \widetilde{q}_\omega^{(j)}, \tag{4.5}$$

where

$$\begin{gathered} \widetilde{A}_\omega^{(j)} = (\widetilde{A}_\omega^{(j),1}, \widetilde{A}_\omega^{(j),2}) = \frac{\omega}{(c^{(j)})^2} v, \\ \widetilde{q}_\omega^{(j)} = q_\omega^{(j)}. \end{gathered} \tag{4.6}$$

Note that the fields $\widetilde{A}_\omega^{(1)}$, $\widetilde{A}_\omega^{(2)}$ do not contain the imaginary part in contrast with $A_\omega^{(1)}$, $A_\omega^{(2)}$.

Now, using (4.3), (4.4), (4.6) and also Theorem 1.1 of [GuT] and simple connectedness of D we get the equalities (3.8), (3.9) for $d = 2$, $\omega \in \{\omega_1, \omega_2\}$, and also the equality

$$\widetilde{A}_\omega^{(2)} - \widetilde{A}_\omega^{(1)} = \nabla \widetilde{\varphi}_\omega \quad \text{in } D, \tag{4.7}$$

where $\widetilde{\varphi}_\omega \in W^{2,\infty}(D, \mathbb{R})$, $\omega \in \{\omega_1, \omega_2\}$.

In a similar way with the proof of Theorem 2.1, proceeding from (3.5), (3.8) for $d = 2$ we obtain (3.14) for $d = 2$.

Now, using (3.4), (4.6) and (4.7) we obtain (3.15), (3.16) for $d = 2$, where

$$\beta_\omega = \widetilde{\varphi}_\omega. \tag{4.8}$$

Finally, using (3.9), (3.14), (3.15), (3.16) for $d = 2$ and (4.8) we complete the proof of Theorem 2.2 in a completely similar way to the corresponding part of the proof of Theorem 2.1.

5 Proofs of Theorems 2.3, 2.4

Note that under the assumptions of Theorems 2.3, 2.4, we have that $L_\omega^{(j)}$ is given by (3.1), where

$$\begin{aligned} A_\omega^{(j)} &= (A_\omega^{(j),1}, \ldots, A_\omega^{(j),d}) = \frac{\omega v^{(j)}}{(c^{(j)})^2} + \frac{i}{2}\frac{\nabla \rho^{(j)}}{\rho^{(j)}}, \\ q_\omega^{(j)} &= -\frac{\omega^2}{(c^{(j)})^2} + i\nabla \cdot \left(\frac{\omega}{(c^{(j)})^2} v^{(j)} + \frac{i}{2}\frac{\nabla\rho^{(j)}}{\rho^{(j)}} \right) - \frac{\omega^2}{(c^{(j)})^4}(v^{(j)})^2 \\ &+ \frac{1}{4}(\rho^{(j)})^{-2}(\nabla\rho^{(j)})^2 - \frac{i\omega v^{(j)}\nabla\rho^{(j)}}{(c^{(j)})^2\rho^{(j)}} - 2i\omega^{1+\zeta^{(j)}}\frac{\alpha_0^{(j)}}{c^{(j)}}, \end{aligned} \tag{5.1}$$

where $j = 1, 2$.

5.1 Proof of Theorem 2.3

In a similar way with the proof of Theorem 2.1, we have formulas (3.3)–(3.7), where in (3.6), (3.7) the functions $q_\omega^{(1)}$, $q_\omega^{(2)}$, $A_\omega^{(1)}$, $A_\omega^{(2)}$ are defined as in (5.1) and $\omega \in \{\omega_1, \omega_2, \omega_3\}$.

Now, separating the real and imaginary parts of (3.6) we obtain equality (3.8) and also the equality

$$\begin{aligned} &\left[\nabla \cdot \left(\frac{v^{(1)}}{(c^{(1)})^2} - \frac{v^{(2)}}{(c^{(2)})^2} \right) - \frac{\nabla\rho^{(1)}}{\rho^{(1)}}\frac{v^{(1)}}{(c^{(1)})^2} + \frac{\nabla\rho^{(2)}}{\rho^{(2)}}\frac{v^{(2)}}{(c^{(2)})^2} \right] \\ &\qquad + 2\omega^{\zeta^{(2)}} \left[\frac{\alpha_0^{(2)}}{(c^{(2)})^2} \right] - 2\omega^{\zeta^{(1)}} \left[\frac{\alpha_0^{(1)}}{(c^{(1)})^2} \right] = 0, \end{aligned} \tag{5.2}$$

where $\omega \in \{\omega_1, \omega_2, \omega_3\}$.

Using (5.2) and the assumption that ω_1, ω_2, ω_3 are positive and mutually different frequencies we obtain, in particular, that (3.9) holds.

In a similar way with the proof of Theorem 2.1, proceeding from (3.4)–(3.9) we obtain (3.21).

Next, in order to show that $\alpha_0^{(2)}(x) = \alpha_0^{(1)}(x)$ for fixed $x \in D$ we consider two cases: (a) $\zeta^{(1)}(x) \neq \zeta^{(2)}(x)$; (b) $\zeta^{(1)}(x) = \zeta^{(2)}(x)$.

For the case (a) using (5.2) and the assumption that ω_1, ω_2, ω_3 are positive and mutually different frequencies, in addition to (3.9), we obtain also that

$$\frac{\alpha_0^{(j)}}{(c^{(j)})^2} = 0 \quad \textit{at point } x, \quad j = 1, 2, \tag{5.3}$$

and, as a corollary,

$$\alpha_0^{(1)}(x) = \alpha_0^{(2)}(x) = 0. \tag{5.4}$$

For the case (b) using (5.2) and the assumption that ω_1, ω_2, ω_3 are positive and mutually different frequencies, in addition to (3.9), we obtain also that

$$\frac{\alpha_0^{(2)}}{(c^{(2)})^2} - \frac{\alpha_0^{(1)}}{(c^{(1)})^2} = 0 \quad \textit{at point } x. \tag{5.5}$$

Using (3.21) and (5.5) we obtain

$$\alpha_0^{(2)}(x) = \alpha_0^{(1)}(x). \tag{5.6}$$

Finally, the result that $\alpha^{(2)} = \alpha^{(1)}$ in D follows from (5.4) for the case (a) and from (5.6) for the case (b).

This completes the proof of Theorem 2.3.

5.2 Proof of Theorem 2.4

In a similar way with the proofs of Theorems 2.2, 2.3 and we have formulas (3.1), (5.1), (3.3)–(3.6) for $d = 2$ and formulas (4.1)–(4.7) where in (3.6), (4.4), (4.7) $\omega \in \{\omega_1, \omega_2, \omega_3\}$.

Now, separating the real and imaginary parts of (3.6) we obtain equality (3.8) and also equality (5.2) for $d = 2$, where $\omega \in \{\omega_1, \omega_2, \omega_3\}$.

Using equality (5.2) and the assumption that ω_1, ω_2, ω_3 are positive mutually different frequencies we obtain, in particular, (3.9) for $d = 2$.

As in the proof of Theorem 2.1, we use (3.5), (3.8) to obtain (3.14).

Using (3.4), (3.9), (3.14) for $d = 2$ and (4.7) as in the proof of Theorem 2.2, we obtain (3.21) for $d = 2$.

Finally, using (5.2) we complete the proof of Theorem 2.4 in a completely similar way to the proof of Theorem 2.3.

6 Proof of Theorem 2.5

Let

$$\mu = \frac{\omega}{c^{(1)}} h. \tag{6.1}$$

One can check by direct computation that

$$e^{-i\mu} L_\omega^{(1)} e^{i\mu} = -\Delta - 2iA_\omega^{(2)}\nabla - U_\omega^{(2)}, \tag{6.2}$$

where

$$\begin{gathered} A_\omega^{(2)} = \frac{\omega}{c^{(1)}}\nabla h, \\ U_\omega^{(2)} = \frac{\omega^2}{(c^{(1)})^2}(1 - |\nabla h|^2) + \frac{2i\omega}{c^{(1)}}(\alpha_0^{(1)} + \tfrac{1}{2}\Delta h), \end{gathered} \tag{6.3}$$

and $e^{i\mu}$, $e^{-i\mu}$ denote the multiplication operators by the functions $e^{i\mu}$, $e^{-i\mu}$, respectively. Using (6.2) one can see that

$$L_\omega^{(2)} = e^{-i\mu} L_\omega^{(1)} e^{i\mu}. \tag{6.4}$$

Due to our assumptions, we have that

$$e^{\pm i\mu} - 1 \textit{ is supported in } D. \tag{6.5}$$

Using (6.4), (6.5) one can see that

$$\sigma(L_z^{(1)}) = \sigma(L_z^{(2)}), \tag{6.6}$$

and

$$\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)} \text{ for all } \omega \in \mathbb{C} \setminus \sigma, \tag{6.7}$$

where $\sigma = \sigma(L_z^{(1)}) = \sigma(L_z^{(2)})$.

This completes the proof of Theorem 2.5.

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